Stochastic equation for a jumping process with long-time correlations

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A jumping process, defined in terms of the jump size and waiting time distributions, is presented. The jumping rate depends on the process value. The process, which is Markovian and stationary, relaxes to an equilibrium and is characterized by a power-law autocorrelation function. Therefore, it can serve as a model of $1/f$ noise as well as of the stochastic force in the generalized Langevin equation. This equation is solved for noise correlations $\sim 1/t$; the resulting velocity distribution has sharply falling tails. The system preserves memory about the initial condition for a very long time.

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I. INTRODUCTION

A jumping process can be defined in terms of two probability distributions which determine the jump size and the waiting time between consecutive jumps. One usually assumes that the two distributions are independent of each other. Such a process is often regarded as a generalized form of the random walk and used to describe diffusive transport. That approach, known as the continuous-time random walk theory [1], is able to account for various forms of diffusion, both normal and anomalous, by a suitable choice of the probability distributions defining the process [2]. Power-law dependences are especially interesting [3,4]. A stochastic trajectory characterized by jump sizes so distributed exhibits a pattern typical for Lévy flights and features systems that reveal enhanced diffusion [5]. On the other hand, long tails of the waiting time distribution (long rests) evoke the opposite effect: they are responsible for subdiffusion [2,6]. Processes that possess such tails are often treated in terms of the fractional diffusion equation [7–10].

For uniform distribution of jumps in time, i.e., if the waiting time probability density has an exponential form, the jumping process relaxes to some stationary equilibrium. The kangaroo process (KP) [11] provides a simple and well-known example. Instead of a jump size distribution, this process assumes a probability distribution of the process value after the jump and, in addition, a jumping rate which depends on the process value. An advantage of the KP from the point of view of possible applications stems from the fact that it can be easily constructed for arbitrary correlations. The need for models of correlated noises is obvious. For example, long correlations, in both space and time, arise as a result of the fast mode removal procedure [12–14]. Long tails of the correlation function emerge also in the relaxation process of a system coupled to a fractal heat bath via a random matrix interaction [15]. In those cases the stochastic dynamics obeys the generalized, non-Markovian, Langevin equation and the Monte Carlo simulation of solutions requires a specific model of the noise. Unfortunately, the KP is not suitable to model noises with power-law correlations: the distribution of the stochastic variable during the trajectory evolution is biased because the waiting time distribution changes its shape when it is inserted into the generalized Langevin equation [16]. As a result, the relaxation to thermal equilibrium cannot be achieved.

In this paper we consider a simple power-law correlated jumping process which is exempt from that difficulty. It can be regarded as a generalization of the KP in which one of the quantities defining the process—the probability density distribution after a jump—has been substituted by the jump size probability distribution. A one-dimensional version of the generalized kangaroo process has been presented in Ref. [17]. The objective of this paper is not only to analyze the master equation for the process but above all to obtain the stochastic variable itself by solving a stochastic equation. Therefore the presented procedure can be utilized as a noise model for numerical simulations of the stochastic trajectories in the framework of the Langevin formalism. We define the process and discuss the corresponding equations in Sec. II. The expression for the autocorrelation function is derived in Sec. III, whereas Sec. IV is devoted to the application of the process as a model of some specific form of the correlated noise in the generalized Langevin equation. The main results are summarized in Sec. V.

II. DEFINITION OF THE PROCESS

We assume that the stochastic process is stepwise, i.e., a process value $\mathbf{x}$ is constant within the time intervals $(t_i, t_{i+1}): \mathbf{x}(t) = \mathbf{x}_i$ for $t \in (t_i, t_{i+1})$. Jumping times t_i are randomly distributed and jumping rate $\nu(\mathbf{x})$ depends on the process value. The size of the jump, defined as the difference between the values of $\mathbf{x}$ after and before the jump, is determined from a given probability distribution $Q(\delta\mathbf{x})$. Then the stochastic trajectory $\mathbf{x}(t)$ obeys the equation

$$\mathbf{x}_{i+1} = \mathbf{x}_i + \delta\mathbf{x} \tag{1}$$

where the waiting time $\tau = t_{i+1} - t_i$ is governed by the Poissonian distribution

$$P_P(\tau) = \nu(\mathbf{x}) e^{-\nu(\mathbf{x})\tau}, \tag{2}$$

which determines the probability density that a jump occurs in the interval $(\tau, \tau + d\tau)$. The initial condition for Eq. (1), $\mathbf{x}(t_0) = \mathbf{x}_0$, follows from the given probability distribution $P_0(\mathbf{x})$. Equation (1) is stochastic because it determines the time evolution of the stochastic variable $\mathbf{x}$, in contrast to the master equation which can give us only probability distribu-

tions. The trajectory $\mathbf{x}(t)$ can be constructed step by step by sampling consecutive values of τ and $\delta\mathbf{x}$ from the distributions P_P and Q, respectively.

The process is Markovian and stationary. The transition probability $p_{tr}d\mathbf{x}$ that the process value is between $\mathbf{x}$ and $\mathbf{x}+d\mathbf{x}$ at an infinitesimal time Δt, providing it was equal to $\mathbf{x}'$ at $t=0$, is given by

$$p_{tr}(\mathbf{x},\Delta t|,\mathbf{x}',0)=[1-\nu(\mathbf{x}')\Delta t]\delta(\mathbf{x}'-\mathbf{x})+\nu(\mathbf{x}')\Delta t Q(\mathbf{x}-\mathbf{x}'). \tag{3}$$

In the above expression we have utilized the fact that p_{tr} may depend only on time differences. The first term on the right-hand side of Eq. (3) is the probability that no jump occurred in the time interval $(0,\Delta t)$. The term $\nu(\mathbf{x}')\Delta t$ means the probability that one jump occurred. The master equation for a probability density $p(\mathbf{x},t)$ can be obtained by calculating the time derivative from $p(\mathbf{x},t)$ and taking into account all possible initial values $\mathbf{x}'$:

$$\frac{\partial}{\partial t}p(\mathbf{x},t) = \lim_{\Delta t\to 0^+}\left(\int p_{tr}(\mathbf{x},\Delta t|\mathbf{x}',0)p(\mathbf{x}',t)d\mathbf{x}'-p(\mathbf{x},t)\right)\Big/\Delta t. \tag{4}$$

We get the master equation in the following form:

$$\frac{\partial}{\partial t}p(\mathbf{x},t)=-\nu(\mathbf{x})p(\mathbf{x},t)+\int Q(\mathbf{x}-\mathbf{x}')\nu(\mathbf{x}')p(\mathbf{x}',t)d\mathbf{x}'. \tag{5}$$

The jumping process described above is still too general and thus we introduce additional restrictions. Let $\mathbf{x}$ be a two-dimensional vector, $\mathbf{x}=(x_1,x_2)$, with the unit length $|\mathbf{x}|=1$. Therefore we require the norm to be preserved during the jumps. With these assumptions, the process can be described in terms of a single angle variable ϕ: $x_1=\cos(\phi)$ and $x_2=\sin(\phi)$. For the probability density Q we take the Gaussian

$$Q(\delta\mathbf{x})\sim e^{-(\mathbf{x}-\mathbf{x}')^2/2\sigma^2}=Ne^{\cos(\phi-\phi')/\sigma^2}, \tag{6}$$

where σ is a given width and the normalization constant $N=1/[2\pi I_0(1/\sigma^2)]$ contains the modified Bessel function. The other quantity defining the process is the jumping rate ν which we assume in the following form:

$$\nu(\phi)=\frac{4}{1-\alpha}\frac{|\sin(\phi)|^\alpha}{|\cos(\phi)|} \tag{7}$$

where $0<\alpha<1$. We will demonstrate in Sec. III that the expression (7) corresponds to a process with power-law autocorrelation function for large times. Taking into account the above assumptions, we obtain the master equation (5) in the one-dimensional form

$$\frac{\partial}{\partial t}p(\phi,t)=-\nu(\phi)p(\phi,t)+\int_0^{2\pi}Q(\phi-\phi')\nu(\phi')p(\phi',t)d\phi'. \tag{8}$$

The equilibrium solution of Eq. (8), $P(\phi)$, has to satisfy the condition $\nu(\phi)P(\phi)=\text{const}$. Therefore, $P(\phi)$ becomes quite simple:

$$P(\phi)=1/\nu(\phi). \tag{9}$$

Since for the jumping rate (7) $\int_0^{2\pi}1/\nu(\phi)d\phi=1$, $P(\phi)$ is properly normalized.

Numerical simulation of stochastic trajectories requires random numbers distributed like $Q(\delta\mathbf{x})$, according to Eq. (6). For that purpose we apply the rejection method which allows us to avoid evaluating complicated integrals. The algorithm is the following. First we sample uniformly distributed random numbers $\delta\phi=\phi-\phi'$ from the interval $(0,2\pi)$. Then $q=Q(\delta\phi)$ is calculated and this value is compared with another random number r_Q, uniformly distributed within the interval $(Q_{\min},Q_{\max})$ where $Q_{\min}$ and $Q_{\max}$ denote the minimum and maximum values of Q, respectively. If $r_Q>q$ then $\delta\phi$ is accepted, otherwise it is rejected and the sampling procedure is repeated.

III. AUTOCORRELATION FUNCTION FOR THE JUMPING PROCESS

The autocorrelation function (ACF) of the process, $\mathcal{C}(t)=\langle\mathbf{x}(0)\mathbf{x}(t)\rangle$, where the average is taken over the stationary distribution $P(\mathbf{x})$, can be evaluated from the following expression [18]:

$$\mathcal{C}(t)=\int\int\mathbf{x}'(t_0)\mathbf{x}(t_0+t)P(\mathbf{x},t|\mathbf{x}')p(\mathbf{x}',t_0)d\mathbf{x}\,d\mathbf{x}'. \tag{10}$$

The conditional probability of passing from $\mathbf{x}'$ to $\mathbf{x}$ during the time t, $P(\mathbf{x},t|\mathbf{x}')$, can be obtained by taking into account all possible paths leading from $\mathbf{x}'$ to $\mathbf{x}$ and summing over the jumps [19]. The final formula for the Laplace transform of the ACF can be expressed by the following series:

$$\begin{aligned}\tilde{\mathcal{C}}(s)=&\int_0^{2\pi}\frac{1}{\nu(\phi)}\frac{1}{\nu(\phi)+s}d\phi\\&+\int_0^{2\pi}\frac{Q(\phi-\phi')}{\nu(\phi)+s}\frac{\cos(\phi-\phi')}{\nu(\phi')+s}d\phi\,d\phi'\\&+\sum_{k=2}^{\infty}\int_0^{2\pi}\frac{\cos(\phi-\phi_0)}{\nu(\phi_0)+s}\frac{Q(\phi-\phi_{k-1})}{\nu(\phi)+s}\\&\times\left(\prod_{i=2}^{k}\frac{Q(\phi_{i-1}-\phi_{i-2})}{\nu(\phi_{i-1})+s}\nu(\phi_{i-1})d\phi_{i-1}\right)d\phi_0 d\phi.\end{aligned} \tag{11}$$

Inverting the Laplace transform we obtain the final expression for the ACF:

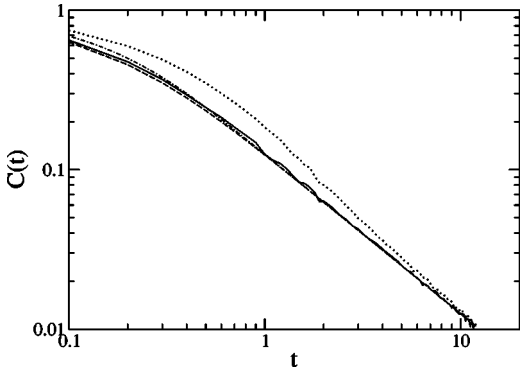

FIG. 1. Autocorrelation function $\mathcal{C}(t)$ for the jumping rate ν given by Eq. (7) with $\alpha=0.5$. Numerical simulations have been performed for $\sigma=2.5$ (solid line) and $\sigma=1$ (dotted line). The first term in Eq. (12) is also shown (dashed line), as well as the parametrization by Eq. (14) (dash-dotted line).

$$\mathcal{C}(t) = 4\int_0^{\pi/2} \frac{e^{-\nu(\phi)t}}{\nu(\phi)} d\phi + 8N\int_0^{\pi/2}\int_0^{\pi/2} (e^{\cos(\phi-\phi')/\sigma^2} - e^{-\cos(\phi-\phi')/\sigma^2}) \times \cos(\phi-\phi') \frac{e^{-\nu(\phi')t} - e^{-\nu(\phi)t}}{\nu(\phi)-\nu(\phi')} d\phi\, d\phi' + \cdots . \tag{12}$$

We are interested in the asymptotic behavior of $\mathcal{C}(t)$ for large t. In this limit the first term of Eq. (12) can be estimated easily. Because of the exponential dependence of the integrand on t, only the vicinity of $\phi=0$ contributes to the integral: $\nu \lesssim 1/t$. Therefore the first term can be approximated by the integral $\int_0^\infty \exp(-\phi^\alpha t)/\phi^\alpha d\phi \sim t^{1-1/\alpha}$. In the second term we first calculate the integral over ϕ: $\int_0^{\pi/2}(\int_0^{\pi/2} d\phi) d\phi'$. If we take the limit of large t in the inner integral, the exponential containing ϕ' can be dropped. Moreover $\nu(\phi)$ becomes negligible, compared to $\nu(\phi')$, as well as ϕ in the arguments of the cosine function. Then for any $\phi'>0$ we have

$$\frac{e^{-\nu(\phi')t} - e^{-\nu(\phi)t}}{\nu(\phi)-\nu(\phi')} \approx e^{-\phi^\alpha t}/\nu(\phi')$$

and the integral over ϕ can be easily evaluated. The required time dependence is of the form $t^{-1/\alpha}$ which means that the second term falls with time faster than the first one. The same conclusion applies to the higher terms. The second term has a simple asymptotic dependence also on the kernel width σ. Expanding the exponential functions over $1/\sigma$ and taking into account that $\lim_{x\to 0} I_0(x)=1$, we find that the second term decreases like $1/\sigma^2$ for large σ. We finally conclude that the ACF can be well approximated by the first term of Eq. (12) and its tail is algebraic:

$$\mathcal{C}(t) \sim t^{1-1/\alpha} \text{ for } t\to\infty. \tag{13}$$

Figure 1 presents ACF for $\alpha=0.5$; $\mathcal{C}(t)$ was calculated from the definition, by means of single trajectory evolution according to Eq. (1), for $\sigma=1$ and $\sigma=2.5$. The equilibrium probability distribution $P(\phi)$ was taken as the initial condition. The result for the larger value of σ agrees very well with the first term in Eq. (12) and it can be parametrized by the function

$$\mathcal{C}(t) = \frac{1-e^{-8t}}{8t}. \tag{14}$$

The existence of long tails of the ACF means that the power spectrum of the process, defined by the Fourier transform $\mathcal{F}(\mathbf{x})$ as $S(\omega)=|\mathcal{F}(\mathbf{x})|^2$, is strongly enhanced at $\omega=0$. The power spectrum can be obtained directly from $\mathcal{C}(t)$ by using the Wiener-Khinchin theorem [18]: $S(\omega)=\mathcal{F}(\mathcal{C}(t))$. For $0.5<\alpha<1$ we get the following result:

$$S(\omega) \sim \omega^{-1/\alpha}. \tag{15}$$

Then our jumping process is characterized by the algebraic power spectrum and becomes $1/f$ noise for $\alpha\to 1$. The overpopulation of small frequency values is due to the fact that the process is dominated by long waiting times between consecutive jumps. Such long intervals correspond to small values of ϕ, i.e., to evolution along the x_1 axis. The quantity $s=1/\nu$, which means the average of the Poissonian distribution (2), is well suited to characterize long rests. The statistics of s is directly connected with the process value probability distribution $P(\phi)$ and, in accordance with that, the density distribution of s in the equilibrium, $\psi(s)$, can be derived from the equation $|\psi(s)ds|=|P(\phi)d\phi|$. In the limit of large s we obtain $\psi(s)\sim s^{-1/\alpha}$ and this result means that a Poissonian waiting time distribution with variable jumping rate can possess, effectively, power-law tails.

The jumping process with $\alpha=0.5$ resembles a deterministic dynamical system: a Lorentz gas of periodically distributed hard disks. In this lattice a particle is elastically reflected by the disks and wanders freely among them. The free paths of the particle are infinite in directions parallel to the symmetry axes. The system is characterized by the velocity autocorrelation function with tail $1/t$, analogously to Eq. (14), and by the power spectrum $S(\omega)\sim|\ln(\omega)|$. However, the long free path distribution falls faster than its stochastic counterpart, as s^{-3} [20].

IV. APPLICATION TO THE GENERALIZED LANGEVIN EQUATION

If a Brownian particle is driven by a stochastic force with a finite correlation time, the time evolution of the velocity obeys the generalized Langevin equation [21,22]

$$m\frac{d\mathbf{v}(t)}{dt} = -\frac{\partial V(\mathbf{r})}{\partial \mathbf{r}} - m\int_0^t K(t-\tau)\mathbf{v}(\tau)d\tau + \mathbf{F}(t) \tag{16}$$

where $V(\mathbf{r})$ is a position-dependent external potential, $\mathbf{F}(t)$ is a stochastic force, and m denotes the mass of the particle. The integro-differential equation (16) can be solved numerically for any $V(\mathbf{r})$ and any memory kernel $K(t)$ if we apply a concrete model for the noise $F(t)$. In the case $V(\mathbf{r})=0$ Eq.

(16) is manageable by Laplace transforms. We obtain the following solution:

$$\mathbf{v}(t) = R(t)\mathbf{v}(0) + m^{-1}\int_0^t R(t-\tau)\mathbf{F}(\tau)d\tau, \tag{17}$$

where the Laplace transform of the resolvent $R(t)$ is given by the equation

$$\tilde{R}(s) = 1/[s + \tilde{K}(s)]. \tag{18}$$

In Eq. (16) the usual damping term—proportional to the velocity—that appears in the ordinary Langevin equation has been substituted by the retarded friction in the form of the memory kernel. In the case $V(r)=0$, Eq. (16) characterizes the equilibrium properly, satisfying the second fluctuation-dissipation theorem [23]. Then the kernel $K(t)$ has to be proportional to the noise ACF $\mathcal{C}(t)$: $K(t)=\mathcal{C}(t)/mk_BT$, where T is the temperature that characterizes the heat bath and k_B is the Boltzmann constant. The introduction of memory friction changes the shape of the velocity autocorrelation function considerably: it is no longer restricted to exponentials.

We wish to demonstrate how the process described in the preceding sections can be applied as a model for the driving stochastic force in Eq. (16). For that purpose we choose an ACF possessing a tail $\sim 1/t$ which characterizes, e.g., noise-induced Stark broadening [24] and nuclear collisions in the framework of a dynamical model [25]. It can also be found in problems connected with phenomena in disordered media [5]. This form of ACF is of special importance for molecular dynamics because it corresponds to the problem of scattering inside a periodic lattice [26]. Let us then consider the ACF given by Eq. (14). Moreover we assume $\langle \mathbf{F}(t)\rangle = 0$. In this case $\tilde{K}(s) = \ln(1+8/s)/8$ and Eq. (18) reads

$$\tilde{R}(s) = \frac{1}{s + \ln(1+8/s)/8}. \tag{19}$$

In order to obtain the resolvent $R(t)$ we need to invert the above transform. Computing the usual contour integral produces the following result:

$$R(t) = e^{-at}(c_1 \sin bt + c_2 \cos bt) - 8\int_0^8 \frac{e^{-tx}dx}{[8x - \ln(8/x-1)]^2 + \pi^2}, \tag{20}$$

where the constants $a=0.3511$, $b=0.2995$, $c_1=0.2297$, and $c_2=1.603$ follow from the numerical evaluation of poles in Eq. (19). The resolvent R has the interpretation of the velocity autocorrelation function $\mathcal{C}_v$,

$$\mathcal{C}_v(t) = \langle \mathbf{v}(0)\mathbf{v}(t)\rangle = \frac{k_BT}{m}R(t). \tag{21}$$

$R(t)$ falls from $R(0)=1$ to negative values and then rises, approaching zero very slowly from below. The behavior of $\mathcal{C}_v(t)$ at $t\to\infty$ is determined by the integral in Eq. (20). In this limit, it becomes simpler: $\sim\int_0^8 e^{-tx}/\ln^2 x\, dx$. Integrating over t yields the integrand in the form $e^{-tx}/(x\ln^2 x)$ and the integral over x can be estimated [19] as $\sim 1/\ln t$. The final expression reads

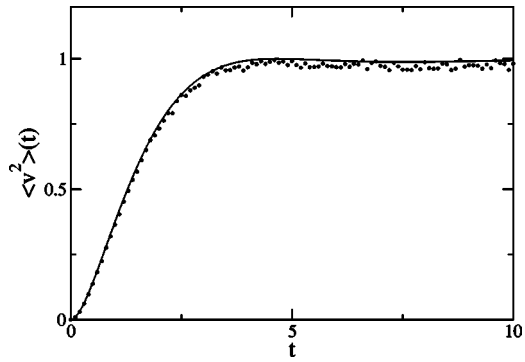

FIG. 2. The velocity variance calculated from Eq. (25) (solid line) and by numerical simulation from Eq. (24) (dots). We assumed $\mathbf{v}(0)=0$, $T=1$, and $m=1$. The units are determined by the condition $k_B=1$.

$$\mathcal{C}_v(t) \sim \frac{-1}{t\ln^2 t} \quad (t\to\infty). \tag{22}$$

Therefore the tail of $\mathcal{C}_v(t)$ diminishes very slowly, like the tail of $\mathcal{C}(t)$, and it is negative.

The velocity autocorrelation function determines the transport properties of the system: the diffusion coefficient can be expressed in terms of the Laplace transform of $\mathcal{C}_v(t)$ in the form $\mathcal{D}=\tilde{\mathcal{C}}_v(s=0)$. Since for $\mathcal{C}(t)$ given by Eq. (14) $\mathcal{D}=0$, the transport is subdiffusive. We come to the same conclusion by the direct calculation of the position variance $\langle \mathbf{r}^2\rangle(t)$. Integrating $\mathcal{C}_v(t)$ twice over time, we get the following estimation:

$$\langle \mathbf{r}^2\rangle(t) \sim \mathrm{li}(t) \approx t/\ln t \quad (t\to\infty). \tag{23}$$

Therefore the deviation from normal diffusion is very small. The same form of the anomalous diffusion has been found in a chaotic (deterministic) system and it has been attributed to intermittency [27,28].

Our aim is to study the motion of the particle by a direct simulation of stochastic trajectories, assuming that the driving force in Eq. (16) is modeled by means of the stochastic process $\mathbf{x}(t)$ and satisfies Eq. (1). We restrict our analysis to the case $V(\mathbf{r})=0$. Inserting the solution of Eq. (1) into Eq. (17) yields the two-dimensional trajectory of the particle velocity:

$$\mathbf{v}(t) = R(t)\mathbf{v}(0) + m^{-1}\left(\mathbf{x}_{n+1}\int_0^{t-t_n} R(\tau)d\tau + \sum_{k=1}^{n}\mathbf{x}_k\int_{t-t_k}^{t-t_{k-1}} R(\tau)d\tau\right), \tag{24}$$

where by sampling of the consecutive jumping times t_k we apply Eq. (2) with $\alpha=0.5$. Moreover, in the following we take the kernel width $\sigma=2.5$. A simple quantity one can evaluate from Eq. (24) is the time dependence of the velocity variance $\langle \mathbf{v}^2(t)\rangle$ where the average is taken over the stationary distribution of the random force (9). Figure 2 presents this quantity, calculated with the initial condition $\mathbf{v}(0)=0$, for $T=1$ and $m=1$. On the other hand, the velocity variance can be derived analytically from Eq. (24); the expression for

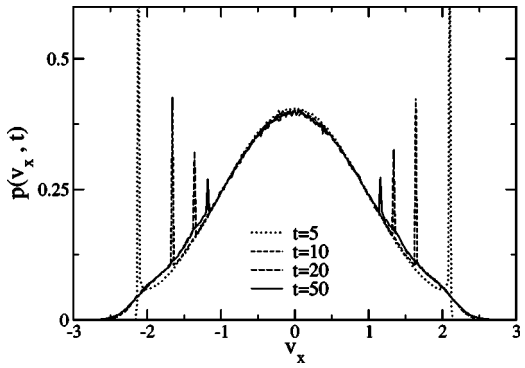

FIG. 3. Time evolution of the probability density distribution of the first velocity component v_x. The stochastic ensemble consists of 5×10^6 trajectories for each time.

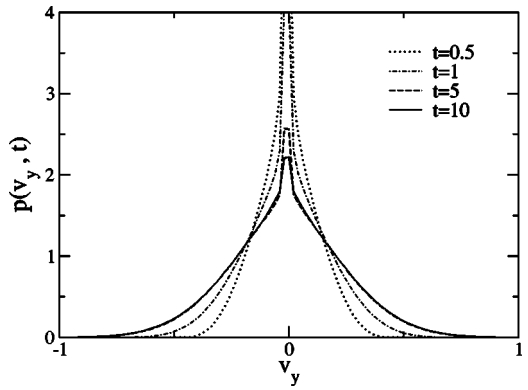

FIG. 4. The same as Fig. 3 except for the second velocity component v_y.

$\langle \mathbf{v}^2(t)\rangle$ involves only the second moment of the noise:

$$\langle \mathbf{v}^2\rangle(t) = m^{-2}\int_0^t\int_0^t R(\tau)R(\tau')\mathcal{C}(|\tau-\tau'|)d\tau\, d\tau'. \quad (25)$$

The velocity variance appears to be independent of a specific noise model and the analytical and numerical results should coincide. Indeed, Fig. 2 demonstrates very good agreement of both results; they indicate relaxation to thermal equilibrium $\langle \mathbf{v}^2\rangle = k_BT/m$ which is apparently reached at about $t=4$ [29].

In a similar way, utilizing Eq. (24), we can determine the density distribution $p(\mathbf{v},t)$ which means the probability that the velocity of the Brownian particle is in the interval $(\mathbf{v},\mathbf{v}+d\mathbf{v})$. Figure 3 presents this distribution, corresponding to the first velocity component v_x, for large times. The central part of the distribution is equilibrated already at $t=5$ but tails are not yet developed; they terminate with high and narrow peaks which originate from the initial condition $p(v_x,0)=\delta(v_x)$. At short times (not shown in the figure) the peaks are still higher and expand gradually with time from the vicinity of the point $v_x=0$. Full relaxation of the tails—which fall off faster than the Gaussian—to the stationary distribution is achieved at $t=20$. Nevertheless, the memory of the initial condition is preserved for a very long time. The distribution of the second velocity component v_y, presented in Fig. 4, looks different; the width is much smaller and the tails show an exponential shape. A complete relaxation to the stationary distribution is reached already at $t=10$. The difference between the distributions for the two velocity components follows from anisotropy of the function $\nu(\phi)$: there are no infinite waiting times corresponding to the motion in the y direction.

The energy spectrum of the Brownian particles deviates considerably from the Maxwellian distribution. Figure 5 presents the time evolution of the probability density distribution of the energy $E=0.5(v_x^2+v_y^2)$. At small values of the energy the curves have a cusp, whereas the tail of the distribution corresponding to the equilibrium state can be parametrized by the function $0.5\exp(-0.5E^2)$ $(E>2)$. It is interesting that the probability density function which characterizes the transport dynamics in the framework of the continuous-time random walk predicts a similar cusp for subdiffusive motion [2].

V. SUMMARY AND DISCUSSION

The jumping process presented in this paper is characterized by the jump size probability distribution and the waiting time distribution, which are mutually correlated. The jumping rate depends on the process value, which is kept constant between consecutive jumps. The process is Markovian and stationary; the corresponding master equation possesses a nontrivial time-independent solution which is completely determined by the jumping rate and which does not depend on the jump size distribution. We have studied the process in its two-dimensional version for jumps that do not change the norm of the process value. An expression for the ACF with power-law tails has been derived. We have demonstrated that it is possible to construct in a simple way a process which is a $1/f$-like noise.

Despite the fact that the waiting time distribution is exponential, the intervals of constant process values can be very long and actually algebraically distributed. This conclusion is not surprising because the mean value of the exponential distribution is also a stochastic variable. Then the existence of long tails of the waiting time distribution does not rule out a relaxation to equilibrium.

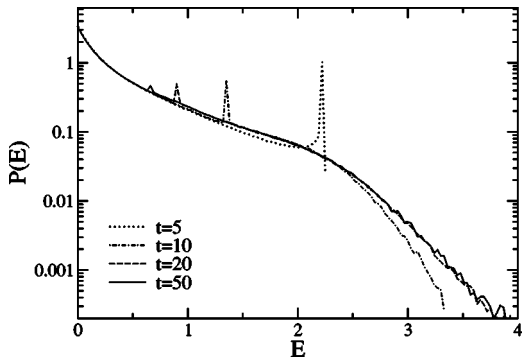

FIG. 5. Time evolution of the probability density distribution of the energy $E=0.5(v_x^2+v_y^2)$.

The considered jumping process resembles the KP because its waiting time distribution also depends on the process value. However, since the probability density Q involves the values both after and before the jump, memory is not lost due to a single jump. In the KP the consecutive jumps are completely independent of each other; the kernel in the master equation factorizes, as well as the conditional probability of passing between arbitrary process values during a given time interval. This feature makes the KP very simple and easily manageable but not very realistic.

The procedure described in this paper allows us to construct stochastic trajectories corresponding to a wide class of power-law ACF's in a simple manner. Therefore it can serve as a model of physical phenomena and can be used as a stochastic force in the generalized Langevin equation. In principle, the KP—which is simpler—can also be used for that purpose. Unfortunately, the generalized Langevin equation solution, simulated in this way, does not relax to thermal equilibrium for power-law distributions [16,30] although the fluctuation-dissipation theorem is satisfied. This apparently paradoxical result follows from the fact that the waiting time distribution changes its shape when $\mathbf{x}(t)$ is evaluated not step by step but at a time t given *a priori*. Such a procedure entails a bias in the process variable distribution which is strengthened by the divergent moments. The stochastic trajectory $\mathbf{x}(t)$ for the process presented in this paper is constructed by sampling jump sizes, i.e., the increments of the process variable, not the variable itself, and thus the above paradox does not appear. Therefore in most cases this process is much better suited than the KP as a model of strongly correlated noises for the generalized Langevin equation.

We have solved this equation for an exemplary form of the ACF, $\sim 1/t$, utilizing our process. Since waiting times are correlated with the direction of the noise vector, the resulting velocity distribution exhibits a strong anisotropy. The distribution of the first component, corresponding to long waiting times, has rapidly falling tails and indicates an extremely long memory about the initial condition, despite the fact that the comprehensive shape of the distribution equilibrates relatively fast. On the other hand, the tails of the distribution corresponding to the second component coincide with the standard Gaussian.

The tail of the ACF is determined predominantly by the long waiting times and thus only one component of the process value is crucial for its shape. Therefore, this component can constitute a one-dimensional counterpart of our two-dimensional jumping process which still has a power-law ACF. This remark is important if one requires a model of noise possessing an arbitrary dimensionality.

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